Toward Synthesizing Numerically Accurate Code for Gauss Pivoting Method: Application to Mechanical Contact Problems

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Introduction :

- We consider a deformable body occupies a bounded domain Ω of $\mathbb{R}^d$ $(d = 1, 2, 3)$
- Γ - boundary of Ω is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3
- Surface tractions of density f_2 act on Γ_2
- Body forces of density f_0 in Ω
- The body is in contact on Γ_3 with a foundation

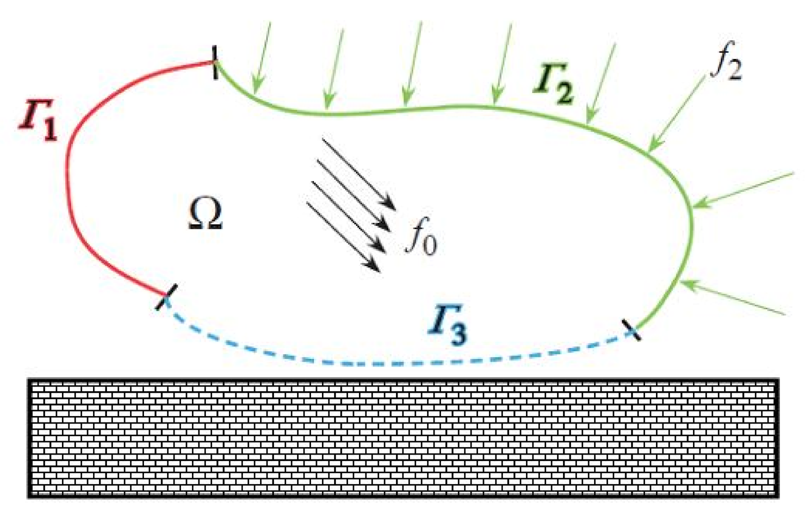


Figure 1: The setting of the problem.

Resolution Steps :

To solve this kind of problems, we need to do the following steps:

- Frictional contact problem formulated with PDE
- Transformation
- Variational problem
- Discretisation
- Approximated variational problem
- by F.E.M or F.D.M
- Non-linear and non-symmetric system
- Newton method
- Solving a linear system for each Newton's iteration

Numerical Accuracy Issues for Gauss :

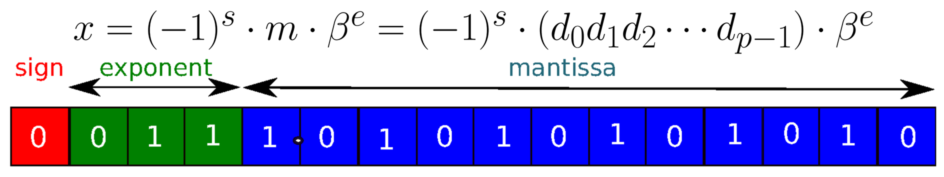
The Gaussian elimination is a process to transform a linear system $Ax = b$ to an upper triangular system $Ux = b'$ that is easy to solve using back substitution method.

Lets take a 3×3 system and lets apply the Gauss pivoting method:

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & -3 \end{pmatrix} x = \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix} \Longrightarrow \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0.333 & -1.67 \\ 0 & 0 & 11.0 \end{pmatrix} x = \begin{pmatrix} 8 \\ -3.67 \\ 22.1 \end{pmatrix} \Longrightarrow x = \begin{pmatrix} 2.01 \\ -0.848 \\ 1.61 \end{pmatrix}, \text{ while } x = \begin{pmatrix} 2.0 \\ -1.0 \\ 1.0 \end{pmatrix}$$

Problem :

$$x = (-1)^s \cdot m \cdot \beta^e = (-1)^s \cdot (d_0 d_1 d_2 \cdots d_{p-1}) \cdot \beta^e$$



- Sensitivity of floating-point arithmetic computations:
  - $(a + b) + c \neq a + (b + c)$
  - $(a + b) \times c \neq a \times c + b \times c$
  - $a + b - b \neq a$
- Floating-point arithmetic not intuitive

```
float x=0.999;
float y = x × x - 2.0 × x + 1.0;
y = 0.000001013278961l8164
float z = (x - 1.0)×(x - 1.0);
z = 0.0000009999743O424992
```

Goal :

To solve these problems, we proceed by synthesizing an accurate and fast program for Gauss pivoting method. We compute with the pairs $([\mathtt{f}], [\mathtt{e}])$ contain:

- $f \in [\mathtt{f}]$ is the floating-point value range,
- $e \in [\mathtt{e}]$ is the error range associated to the floating-point interval $[\mathtt{f}]$

The Rock-N-Roll Tool :

- **Input:**

$$[\mathtt{S}] : \begin{pmatrix} ([\mathtt{f}]_{11}, [\mathtt{e}]_{11}) & \cdots & ([\mathtt{f}]_{1n}, [\mathtt{e}]_{1n}) \\ \vdots & \ddots & \\ ([\mathtt{f}]_{n1}, [\mathtt{e}]_{n1}) & \cdots & ([\mathtt{f}]_{nn}, [\mathtt{e}]_{nn}) \end{pmatrix} \mathtt{x} = \begin{pmatrix} ([\mathtt{f}]_1, [\mathtt{e}]_1) \\ \vdots \\ ([\mathtt{f}]_n, [\mathtt{e}]_n) \end{pmatrix}$$

- **Output:** `C-program` to compute the solution `x`
- **Algorithm:**

1. `[S] ⟵GaussSynthesis([A],[b]);`
2. `x ⟵BackSubSynthesis([A],[b]);`
3. print the `C-program`

Numerical Experimentations :

Case 1D: Flexion of a Beam

We consider an elastic beam fixed on its extremities as follows :

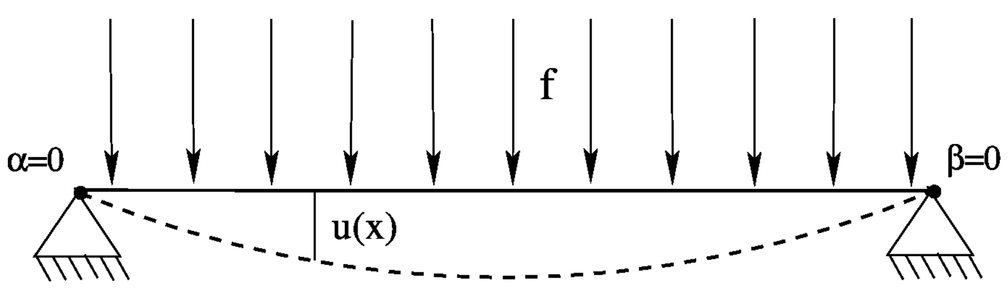


Figure 2: Physical setting of the flexion of a 1D beam.

Problem $\mathcal{P}$: Find a displacement field $u \in C^2([0,1], \mathbb{R})$ such that,

$$\begin{cases} -u''(x) = f & \forall x \in]0,1[\\ u(0) = \alpha \quad \text{and} & u(1) = \beta, \end{cases}$$

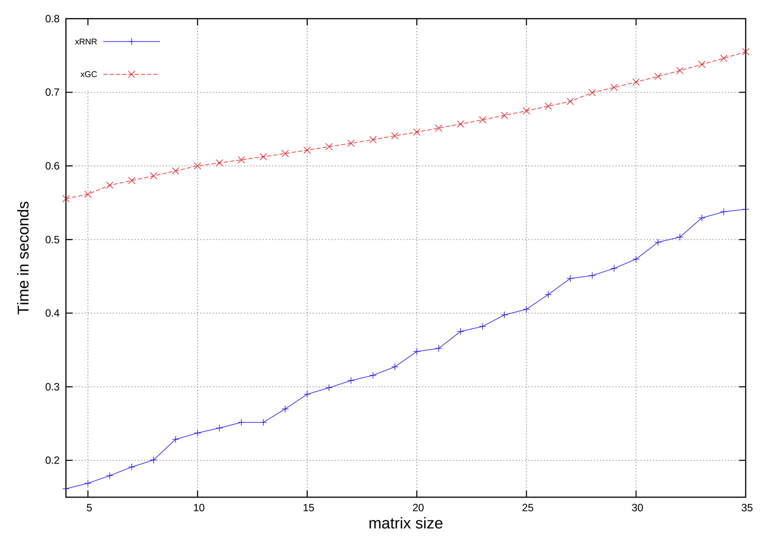


Figure 3: $Cond(A) = 1/\varepsilon = 10^6$: Execution Time.

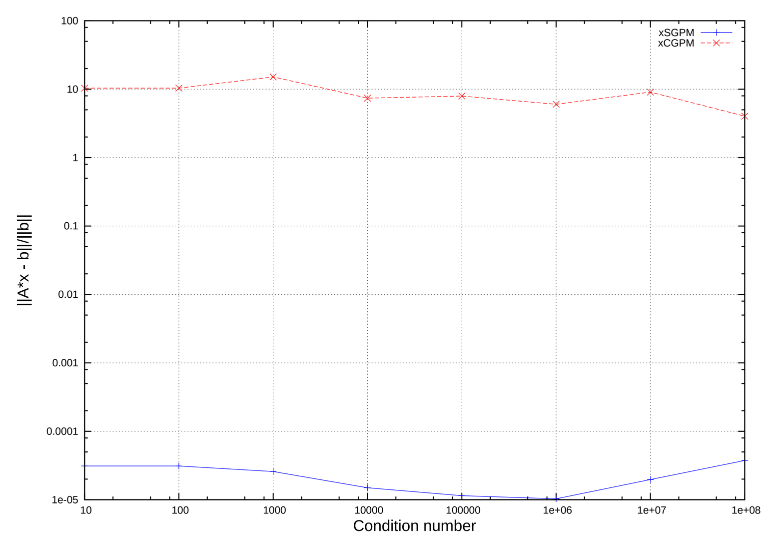


Figure 4: $N = 40$: Relative Error

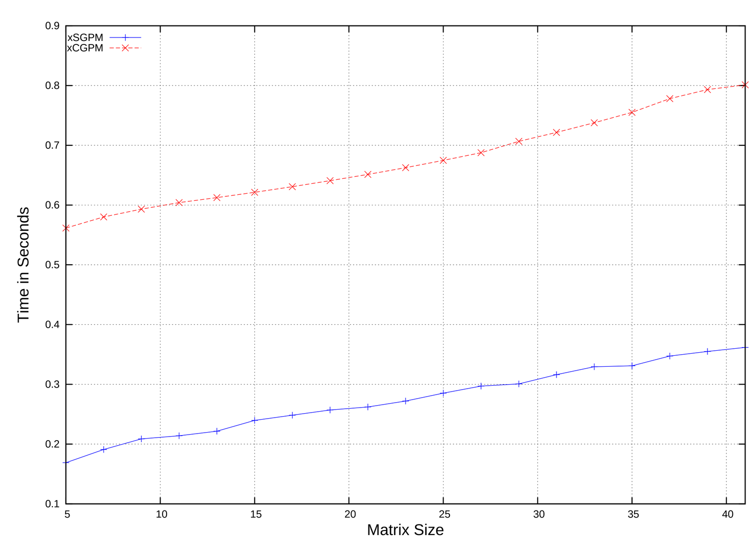


Figure 5: $N = 40$: Execution Time

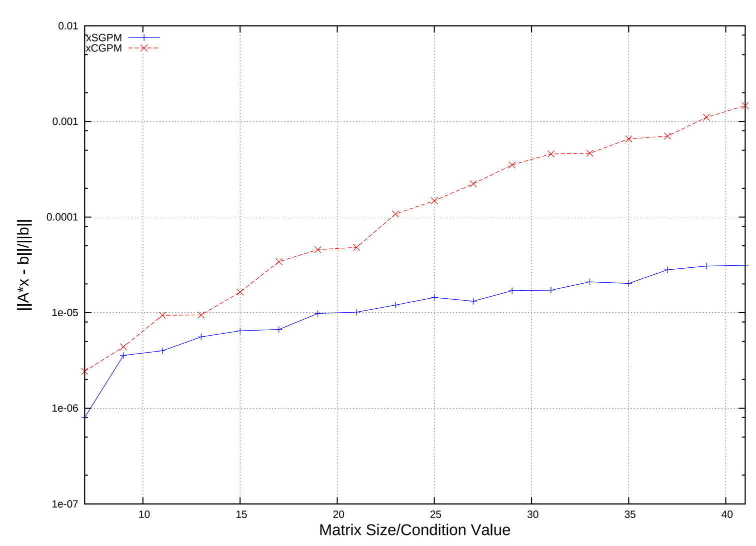


Figure 6: $Cond(A) = N$

Case 2D: A frictional contact problem with a moving foundation :

We consider a non-trivial problem which describes the sliding contact of a 2D viscoelastic body against a moving foundation.

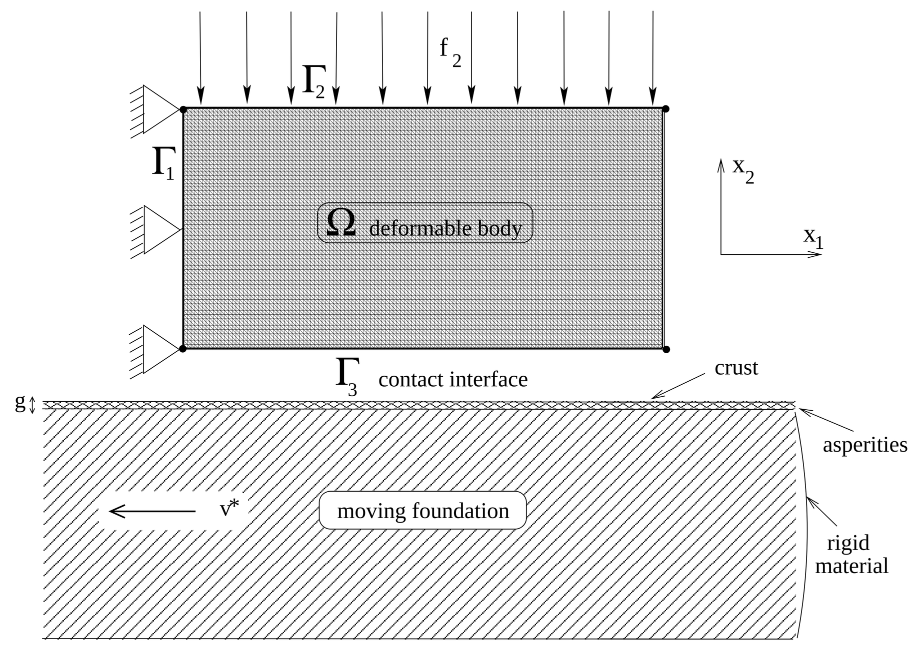


Figure 7: The setting of the problem.

Problem $\mathcal{P}$: Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that, for all $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{M}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in } \Omega, \quad (1)$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \quad (2)$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \quad (3)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \quad (4)$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu|\mathcal{R}\sigma_\nu(t)|\,\boldsymbol{\tau}^* \quad \text{on } \Gamma_3, \quad (5)$$

and there exists a normal reaction $\pi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies

$$\left.\begin{array}{l} u_\nu(t) \le g, \sigma_\nu(t) + p(u_\nu(t)) + \pi(t) \le 0, \\ (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \pi(t)\big) = 0, \\ 0 \le \pi(t) \le F\Big(\int_0^t \mathcal{N}(t-s)u_\nu^+(s)\,ds\Big), \\ \pi(t) = 0 \text{ if } u_\nu(t) < 0, \\ \pi(t) = F\Big(\int_0^t \mathcal{N}(t-s)u_\nu^+(s)\,ds\Big) \text{if } u_\nu(t) > 0 \end{array}\right\} \text{ on } \Gamma_3. \quad (7)$$

The relative errors:

matrix size	10	16	22	28
$rel(x_{SGPM})$	1.33974e-08	5.99157e-07	4.41512e-07	1.67206e-07
$rel(x_{CGPM})$	8.02164e-06	9.12376e-05	5.62905e-05	3.55948e-05

The execution time:

matrix size	10	16	22	28
SGPM	0.252621	0.261653	0.266668	0.278529
CGPM	0.328312	0.349035	0.363256	0.410162

Perspectives :

- Integrate matrix partitioning to handle size problem,
- Synthesize an iterative method : Conjugated Gradient,
- Integrate parallel computing to our tool,
- Study the impact on Newton convergence,
- Apply the tool to novel problems.

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